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Approved by R. F. Brown Director / Underwater Acoustics Division

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RESUMÉ

Une équation simple et générale des vitesses de groupe des modes normaux donne un aperçu des facteurs qui déterminent la vitesse de groupe, notamment dans les eaux peu profondes en forte interaction avec le fond marin. La présente communication en fait la démonstration à l'aide d'un modèle théorique d'un haut-fond, en comparant des formes de mode a différentes fréquences. L'équation se prête particulièrement bien à l'utilisation de codes informatiques de modes normaux qui permettent déjà d'effectuer des calculs semblables de modes normalisés et de coefficients d'atténuation. Fondée sur l'expression approchée WKB des fonctions des modes normaux, l'équation donne la vitesse horizontale moyenne du rayon équivalent au mode. Ce résultat renforce l'analogie rayon-mode utilisée dans l'analyse des problèmes d'acoustique en milieu marin.

The group velocity of normal modes

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A simple, general formula for normal mode group velocities provides an intuitive grasp of the factors influencing group velocity, especially for shallow water environments associated with strong seabed interaction. This is demonstrated using some hypothetical shallow water environments, and by comparing mode shapes at different frequencies. The formula is especially convenient to implement in normal mode computer codes which already perform similar calculations for mode normalization and attenuation coefficients. Using the WKB approximation to the normal mode functions, the formula is identified with the average horizontal speed of the ray equivalent to the mode. For a shallow water environment with bottom interaction, the group velocity formula is shown to include the beam and the time displacement effects of modified ray theory. This result strengthens the ray/mode analogy which is used in analyzing ocean acoustics problems.

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INTRODUCTION

A useful representation of the acoustic field in shallow water is the normal mode representation, in which the distinct modes $n = 1, 2, 3, \ldots N$ (i.e., distinct vertical distributions of pressure) travel along the shallow water waveguide with the phase velocities $c_n = \omega/k_n$, where ω is the angular frequency of a sinusoidal source, and k_n is the longitudinal wavenumber of the nth mode. For an impulsive signal, the speed of energy transport in a given mode n at a frequency ω is given by the group velocity $v_g^{(n)} = |d\omega/dk_n|$, evaluated at ω. The group velocity is often calculated from theory using a finite-difference estimate of this derivative, but one can derive an expression for the modal group velocity in terms of an integral representing a weighted average of the sound-speed profile.

This formula has previously been derived by Tolstoy[1] using considerations of energy flux and energy density. An alternative approach, reported by Koch *et al.*,[2] makes use of the mathematical properties of the mode functions and the differential equation which governs them, to obtain the same expression from the definition $v_g^{(n)} = |d\omega/dk_n|$. These authors included shear wave effects in their derivation. A simplified form of this derivation, for use in continuously stratified fluid media, is presented in the Appendix to this paper.

As will be demonstrated in Sec. I, this formula provides an intuitive grasp of how the sound-speed profile shape affects the resultant group velocity, whereas the fundamental definition $d\omega/dk_n$ obscures this relationship. Since the formula is perfectly general with respect to sound speed and density profiles, it is capable of accomodating shallow water conditions, in which large discontinuities may occur. By employing the formula in some hypothetical shallow water environments, and by comparing the normal mode functions at several frequencies, the shape of a typical group velocity curve can be explained qualitatively.

It is also possible to treat shallow water problems using ray theory.[3] In the context of this paper, the propagation of an impulse of energy in several modes with different group velocities has an analog in ray theory: the energy is viewed to travel along multiple ray paths having different travel times. The elements of this ray/mode analogy are reviewed in Sec. II, and the average horizontal speed of a ray is calculated in Sec. III. In Sec. IV, the WKB method is used to demonstrate the correspondence between the normal mode group velocity and the equivalent average ray speed for a deep ocean profile. In Sec. V, bottom interaction effects in shallow water are shown to generate beam and time displacements in ray theory. These results extend the ray/mode analogy which is so useful in explaining acoustic propagation in the ocean.

I. THE GROUP VELOCITY OF A NORMAL MODE

For the ocean acoustics problem, the normal modes $U_n(z)$ with wavenumbers k_n are the solutions to the depth equation[4]

$$\rho(z)\frac{d}{dz}\left(\rho^{-1}(z)\frac{dU_n(z)}{dz}\right)+\left(\frac{\omega^2}{c^2(z)}-k_n^2\right)U_n(z)=0, \tag{1}$$

with the boundary conditions

$$U_n(0)=0, \tag{2a}$$

$$\lim_{z\to\infty} U_n(z)=0. \tag{2b}$$

The density $\rho(z)$ appears in this way to accomodate gradients or discontinuities in the medium while leaving the pressure $U_n(z)$ and particle velocity $\rho^{-1}(z)U_n(z)$ continuous. Each mode $U_n(z)$ represents a vertical distribution of pressure at the frequency ω propagating as a cylindrically spreading wave with the unique phase velocity $c_n = \omega/k_n$. The full solution to the wave equation at a single frequency is a sum of normal modes, each multiplied by the Hankel function $H_0^{(1)}(k_n r)$ which supplies the range dependence. For impulsive sources, the waveform is not harmonic, but is composed of a band of frequencies. Each frequency component has a slightly different set of normal modes and wavenumbers.

The speed of energy transmission in a given mode is determined by the interference between the various frequency components, and is termed the group velocity, given by the expression:

$$v_g^{(n)} = \left| \frac{d\omega}{dk_n} \right|. \tag{3}$$

Each mode has a distinct group velocity, which varies with frequency. This fact accounts for the time spreading of an impulsive signal in a waveguide. Note that this dispersion is due to the guiding of energy in the region of minimum sound speed, caused by reflection at boundaries or refraction by the sound-speed profile, and is not due to any intrinsic dispersion of the medium. This topic is discussed by Clay and Medwin[4] and Brillouin.[5]

Expression (3) leads to an analytic formula for group velocity only for those cases admitting normal mode functions and wavenumbers in analytic form. Even the simple two-layer fluid model requires numerical solution of a characteristic equation to provide wavenumbers, which then can be substituted into an expression for group velocity. The group velocity may be estimated in other cases by solving expression (1) at two frequencies $\omega + \Delta\omega$ to find wavenumbers with the difference $\Delta k_n = k_n(\omega + \Delta\omega) - k_n(\omega - \Delta\omega)$, and forming the finite-difference estimate

$$\hat{v}_g^{(n)} = |\Delta\omega/\Delta k_n|. \tag{4}$$

This technique is subject to numerical error, and does not provide any understanding of what factors influence normal mode group velocities. Tolstoy[1] and Koch *et al.*[2] have presented an elegant alternative to this method. Their approach makes use of the mathematical properties of the normal modes inherent in Eqs. (1), (2a), and (2b), and provides the general formula

$$v_g^{(n)} = \frac{1}{c_n}\left(\int_0^\infty dz \frac{1}{\rho(z)} U_n^2(z) \Big/ \int_0^\infty dz \frac{1}{\rho(z)c^2(z)} U_n^2(z)\right). \tag{5}$$

This expression, derived in the Appendix, is evidently some kind of weighted average of the sound-speed profile. The type of averaging process it resembles is known as a harmonic average. As an example, the net resistance of a group of N resistors wired in parallel is equal to the harmonic average of the individual resistances, divided by N. A more relevant example is that of a particle moving along a trajectory made up of legs of different lengths with a different speed along each leg. The average speed of the particle is the harmonic average of the speed on each leg, weighted by the leg distances. For a layered fluid with piecewise constant properties, Eq. (5) could be stated: the product of the group and phase velocities is equal to the weighted harmonic average of the squares of the layer sound speeds. The weighting coefficients are given by the relative amounts of energy propagating in each fluid layer. In this way the expression (5) is reminiscent of calculations in quantum mechanics, in which $\rho^{-1}(z)U_n^2(z)$ plays the role of a probability density, and $c^{-2}(z)$ an observable whose expectation value is required. The integrals in this expression also appear similar to those used to compute normal mode normalization constants and bottom attenuation coefficients.[6] This fact suggests that implementation of a group velocity calculation using (5) would be a straightforward addition to existing computer codes, which routinely evaluate integrals of this sort.

The effect of the mode shape on group velocity can be demonstrated using a three-layer shallow water model consisting of 100 m of water ($c = 1500$ m/s, $\rho = 1$ g/cm^3), 100 m of "silt" ($c = 1650$ m/s, $\rho = 1$ g/cm^3), and an infinite half-space of "rock" ($c = 4500$ m/s, $\rho = 2$ g/cm^3). For comparison with this model, two different two-layer models were devised, one consisting of 100 m of water and an infinite half-space of silt, and the other consisting of 200 m of water and an infinite half-space of rock. For these results, shear effects in the rock have been ignored. The parameters of these three geoacoustic models are summarized in Fig. 1.

These models were used as input to the normal mode computer program PROLOS[7] to calculate wavenumbers k_n and mode functions $U_n(z)$ over a range of frequencies. A small utility program was written to evaluate the group velocity at these frequencies by a numerical integration of expression (5). The resulting group velocity curves for mode $n = 3$ for all three models of Fig. 1 are shown in Fig. 2. The crosses mark frequencies at which the eigenfunction of mode 3 was calculated. These mode functions are plotted in Fig. 3. Note that the mode amplitude, representing the acoustic pressure, is continuous across the boundaries. As an aid in gauging the effect of the three layers at each frequency, we have calculated the percentage contribution of each layer to the group velocity integral and to the normalization integral, which is the numerator of Eq. (5). These quantities appear in Table I along with the corresponding phase and group velocities.

At frequencies above 50 Hz, the penetration of the mode into the rock layer is apparently negligible, so the group velocity curve for the three-layer model is virtually identical to the group velocity curve for two-layer model A. At frequencies below the mode cutoff for this two-layer model, the mode is oscillatory in the silt layer of the three-layer model, and begins to penetrate the rock layer. At 37 Hz, there is a resonance in the silt layer, in the sense that this layer's contributions to the normalization and group velocity calculations are 87.1% and 85.2%, respectively. Here the group velocity curve has a local maximum of 1587 m/s. This

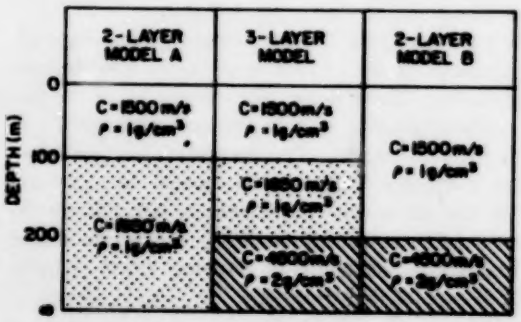

FIG. 1. Parameters of the hypothetical shallow water models; c is sound speed and ρ is density.

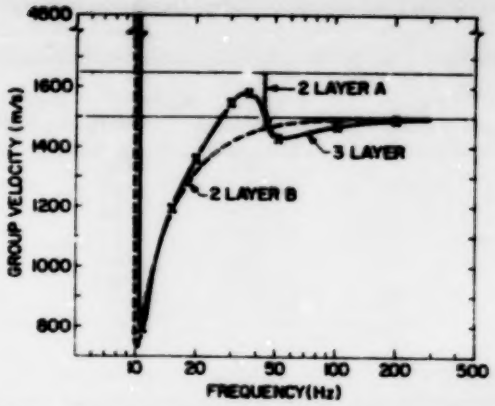

FIG. 2. Group velocity versus frequency for mode 3 for the three hypothetical models. Symbols refer to mode plots in Fig. 3. The curve for two-layer model A is virtually identical with that for the three-layer model above 50 Hz.

is not unrelated to the fact that the phase of the mode function advances through approximately 180° in this layer. (The resonance effect is more pronounced for higher modes, as shown by Tolstoy.[1]) At still lower frequencies, the group velocity decreases towards an absolute minimum. In this region the mode function is not very sensitive to the impedance contrast between the water and silt layers. This is reflected in the roughly equal contribution of these layers to the calculations. The three-layer curve looks similar to two-layer curve B here.

As frequency is lowered, the general increase of phase velocity and decrease in group velocity of a given mode may seem curious, but this effect can be understood by imagining the mode to be the interference pattern created by two plane waves, one traveling upward at a given angle to the horizontal, and one traveling downward at the same angle. To maintain the same number of interference nulls (i.e., the mode number) as frequency is decreased, the angle of propagation of these waves must steepen. The projection of the wave vector in the horizontal direction decreases, so the phase velocity must increase. The energy spends more time traveling up and down the waveguide, so the net horizontal speed of energy transport (the group velocity) decreases. This concept of the equivalence between modes and rays will be expanded in later sections. The existence of a minimum in the group velocity, followed by a rapid increase to the sound speed in rock at mode cutoff is very much a wave phenomenon and is explained by the dramatic increase in the weight of the rock layer at these low frequencies (cf. 10.55 Hz in Table II). To explain this effect using a ray approach would require a modified theory which includes beam and time displacements (Sec. V).

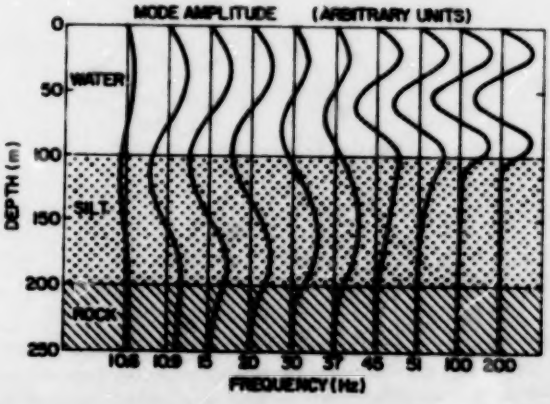

FIG. 3. Amplitude of mode 3 as a function of depth for the three-layer model at several frequencies of interest.

The local maximum in the group velocity of mode three has been attributed to a resonance in the silt layer. For this environment, only one such resonance is possible for this mode, and it has a large width. The effect is more pronounced for higher mode numbers, and the number of resonances may increase. For example, mode 5 in the same environment has a strong resonance at 73 Hz, and a weaker resonance at 50 Hz, and three local minima in the group velocity curve, shown in Fig. 4. The mode functions are plotted in Fig. 5, with preference given to frequencies of extreme in the group velocity curve. The corresponding phase and group velocities are given in Table II, along with the percentage contributions of the layers to the normalization and group velocity integrals. In this case, the resonance at 73 Hz is more apparent, the silt layer contributing 95.3% of the normalization and 94.5% of the group velocity.

II. THE RAY/MODE ANALOGY

There is a wealth of literature[3,8-18] on the equivalence between the normal mode representation and the ray (geometric-optics) representation of the underwater acoustics propagation problem. In particular, Guthrie and Tindle[12] have demonstrated that the interference pattern created by a group of adjacent normal modes propagating in the ocean waveguide can be identified with an equivalent ray trajectory. The relationship between the wavenumber k_n of the central mode of the group at frequency ω, and the angle from the horizontal $\theta_n(z)$ of the equivalent ray in a stratified ocean characterized by the sound-speed profile $c(z)$ is

$$c(z)/\cos\theta_n(z) = \omega/k_n \equiv c_n, \tag{6}$$

in which c_n is the phase velocity of the nth mode. The ray trajectory can be traced from expression (6) (Snell's law). The turning points of the ray [at which $\theta_n(z) = 0$] are found to occur when $c(z) = c_n$. The skip distance D_n of the equivalent ray (the horizontal distance between two consecutive turning points at the same depth) is related to the spacing of the eigenvalues k_n at the center of the mode group:

$$D_n = -2\pi\left(\frac{\partial n}{\partial k_n}\right)_\omega. \tag{7}$$

This relation arises from the condition that D_n is the distance over which the phase difference between two adjacent modes advances by 2π. In other words, D_n is the spacing of the peaks in the interference pattern created by the adjacent modes. Since n is an integer and the spectrum of wavenumbers is discrete, one would evaluate (7) using the finite difference $\Delta n/\Delta k_n$, which is a smoothly changing

TABLE I. Mode 3 layer contributions. % Norm and % V_g refer to the percentage contribution of the layer to the calculation of the normalization and group velocity integrals, respectively.

Frequency (Hz)	Phase velocity (m/s)	Group velocity (m/s)	Water		Silt		Rock	
			% Norm	% V_g	% Norm	% V_g	% Norm	% V_g
10.55	4088.0	3916.3	0.9	7.0	1.2	7.8	97.9	85.2
10.9	3982.9	790.8	32.7	45.8	43.7	50.5	23.6	3.7
15.0	2192.6	1182.9	43.5	50.1	51.7	49.3	4.8	0.6
20.0	1660.8	1354.1	47.4	53.1	50.4	46.6	2.2	0.3
30.0	1702.2	1548.8	19.0	22.0	80.2	77.7	0.8	0.1
37.0	1676.7	1587.0	12.5	14.8	87.1	85.2	0.3	0.05
45.0	1652.8	1482.9	52.9	57.6	47.1	42.4	...	...
51.0	1625.8	1432.5	80.4	83.3	19.5	16.7	...	...
100.0	1535.0	1470.0	98.4	98.7	1.6	1.3	...	...
200.0	1509.0	1491.5	99.8	99.8	0.2	0.2	...	...

"function" of k_n for high mode densities. Nevertheless, it is often convenient to regard these parameters as continuous real variables to perform theoretical manipulations.

Figure 6 is a schematic illustration of a normal mode and its equivalent ray, identifying the parameters of interest.

Equations (6) and (7) are the basis of the analogy between rays and modes used to explain problems in underwater acoustics. As well as illustrating ray effects in normal mode theory,[10–12] the analogy has been used to relate mode and ray cycle distances, bottom reflection loss, and mode attenuation.[14–17] The remainder of this paper will discuss the correspondence between normal mode theory and ray theory as they relate to group velocity. This correspondence has been demonstrated by Guthrie[10] for the deep ocean sound channel, using a finite difference calculation of mode group velocity. For a shallow water environment described by a simple two-layer model, Tindle and Bold[18] have demonstrated the equivalence both experimentally and analytically. We will use the WKB approximation to demonstrate this correspondence in both deep and shallow water with arbitrary sound-speed profiles.

III. TRAVEL TIME ALONG A RAY TRAJECTORY

It has already been stated that the group velocity properly describes the speed of energy transport in the normal mode representation. In the ray representation, the horizontal speed of an impulsive signal over a long range (i.e., many ray cycles) is given by the average horizontal ray speed, which is the horizontal distance between turning points divided by the time-of-flight between turning points. This averaging is necessary because the phase speed and horizontal angle of the ray vary along the trajectory between turning points.

In the ray representation, the speed of energy transmission along a ray is equal to the phase velocity $c(z)$, since the field is locally approximated by a plane wave, and the medium is assumed to be nondispersive (ignoring volume attenuation). Due to refraction, the ray traces a curvilinear trajectory up and down the water column, the phase velocity $c(z)$ and horizontal angle $\theta(z)$ vary along the trajectory, and consequently the local horizontal component of velocity varies along the trajectory. In the case of sound channel propagation, the ray repeats its pattern of motion between the turning points, and completes many cycles over a long range path, so the average horizontal velocity (range divided by travel time) will be practically the same as the horizontal velocity averaged between the upper and lower turning points. Referring again to Fig. 6, this velocity is

$$\langle v \rangle = \int_{\circlearrowright} dx(z) \Big/ \int_{\circlearrowright} dt(z) \tag{8}$$

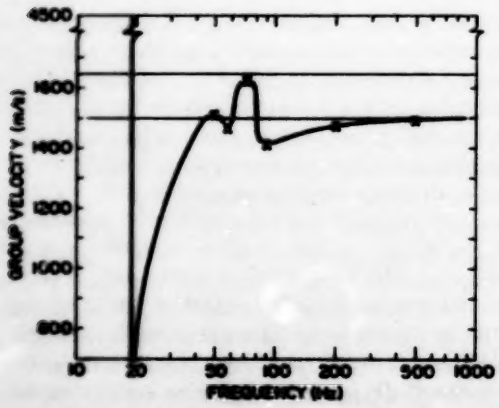

FIG. 4. Group velocity versus frequency for mode 5 for the three-layer model. Symbols refer to mode plots in Fig. 5.

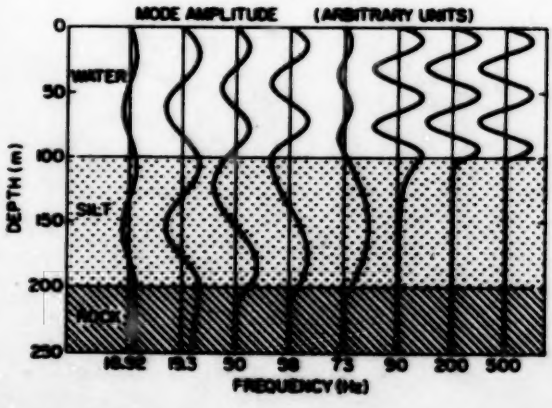

FIG. 5. Amplitude of mode 5 as a function of depth for the three layer model.

TABLE II. Mode 5 layer contributions. % Norm and % V_g refer to the percentage contribution of the layer to the normalization and group velocity integrals, respectively.

Frequency (Hz)	Phase velocity (m/s)	Group velocity (m/s)	Water		Silt		Rock	
			% Norm	% V_g	% Norm	% V_g	% Norm	% V_g
18.92	4500.0	2845.0	5.3	26.2	6.0	24.8	88.7	49.0
19.3	4115.0	709.7	38.8	50.3	44.0	47.2	17.2	2.5
20.0	3543.1	766.2	42.4	51.2	47.6	47.4	10.0	1.3
50.0	1717.0	1512.8	25.1	29.0	74.4	70.9	0.5	0.1
58.0	1682.2	1472.5	47.9	52.7	51.9	47.3	0.2	0.0
73.0	1699.0	1[illegible]6.7	4.6	5.5	95.3	94.5	0.1	0.0
90.0	1625.5	1[illegible]4.3	87.7	89.5	12.3	10.4	...	...
200.0	1525.5	1476.4	99.4	99.5	0.6	0.5	...	...
500.0	1504.1	1496.0	100.0	100.0	...	...	...	...

in which a and b are the upper and lower turning points. This can be converted into depth integrals using Snell's law:

$$c(z)/\cos\theta(z) = c_t, \tag{9}$$

where c_t is the sound speed at the turning points. From this, we find the relations

$$dt = ds/c(z) = dz\, c^{-1}(z)\csc\theta(z), \tag{10a}$$

$$dx = dz\cot\theta(z), \tag{10b}$$

where ds is the arc length along the ray. Now expression (8) can be written as

$$\langle v\rangle = c_t^{-1}\int_a^b dz\, c(z)\csc\theta(z) \Big/ \int_a^b dz\, c^{-1}(z)\csc\theta(z), \tag{11}$$

in which

$$\csc\theta(z) = [1 - c^2(z)/c_t^2]^{-1/2}. \tag{12}$$

Expression (11) will next be compared with the normal mode group velocity.

IV. THE CORRESPONDENCE BETWEEN RAY THEORY AND WAVE THEORY: DEEP WATER

It is clear that expression (5) for the normal mode group velocity is a mode-dependent average of the sound-speed profile, and must be the wave theoretic analog to the average horizontal speed of the equivalent ray calculated from expression (12). The correspondence between these two can be shown by employing the WKB[19] approximation to the normal mode functions in (5). This substitution gives the high-frequency limit of the wave theoretic result. The WKB modes between the ray turning points in a sound channel formed by a continuously varying sound-speed profile are

$$\hat{U}_n(z) = A_n(z)\sin\phi_n(z) \tag{13}$$

in which $A_n(z)$ is the slowly varying amplitude

$$A_n(z) = \gamma_n^{-1/2}(z) = [\omega^2/c^2(z) - k_n^2]^{-1/4} \tag{14}$$

and the WKB phase is

$$\phi_n(z) = \phi_a + \int_a^z dz'\,\gamma_n(z'). \tag{15}$$

The quantity ϕ_a is the phase at the upper turning point, and depends on the boundary condition, but is unimportant for this discussion. We assume a constant density and a slowly varying sound speed. If (13) is substituted into (5), the rapidly oscillating $\sin^2\phi$ factor can be replaced by its average value. The contribution from the exponentially decaying mode tails outside the turning points is assumed to be negligible, so that

$$v_g^{(n)} \simeq \left(\frac{k_n}{\omega}\right)\int_a^b dz\,\gamma_n^{-1}(z) \Big/ \int_a^b dz\, c^{-2}(z)\gamma_n^{-1}(z). \tag{16}$$

The concept of the equivalent ray from (6) allows the substitution

$$\gamma_n(z) = [\omega/c(z)]\sin\theta_n(z) \tag{17}$$

so that

$$v_g^{(n)} \simeq c_n^{-1}\int_a^b dz\, c(z)\csc\theta_n(z)\left(\int_a^b dz\, c^{-1}(z)\csc\theta_n(z)\right)^{-1}. \tag{18}$$

This is identical with expression (12) for the average horizontal ray speed, recalling that c_n is the speed at the turning points of the ray equivalent to the nth mode. This result establishes the correspondence of ray theory and wave theory as they relate to ray travel time per cycle and mode group velocity for a deep ocean sound channel, at least for cases in which the WKB approximation is valid. Of course, expression (5) was derived from the definition of group velocity using no approximations, and is exact. In particular, it rou-

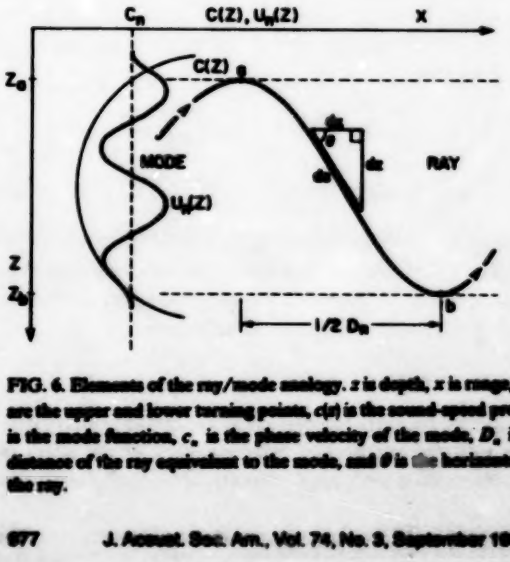

FIG. 6. Elements of the ray/mode analogy. z is depth, x is range, z_a and z_b are the upper and lower turning points, $c(z)$ is the sound-speed profile, $U_n(z)$ is the mode function, c_n is the phase velocity of the mode, D_n is the skip distance of the ray equivalent to the mode, and θ is the horizontal angle of the ray.

tinely includes bottom interaction effects, whereas ray theory must be modified to include these effects, as will be shown in the next section.

V. THE CORRESPONDENCE BETWEEN RAY THEORY AND WAVE THEORY: SHALLOW WATER

It is interesting to compare the different manner in which ray theory treats bottom interaction effects as opposed to normal mode theory. The latter method regards the seabed as part of the medium, and a mode trapped in the water actually penetrates to some extent into the bottom, within which a proportion of the energy in the mode travels horizontally. The relationship between mode shape and group velocity was explored in Sec. I. On the other hand, previous authors[15–18] have shown that ray theory must be modified by the addition of terms to the ray cycle distance and travel time to account for bottom effects. The beam displacement concept was used by Murphy and Davis[13] to account for the effect of a nearby boundary (the sea surface) on the phase shift of a ray refracted at a turning point. The emphasis shifted to bottom effects[14–18] when it became clear that beam displacement was necessary to reconcile ray reflection loss per ray skip distance with normal mode attenuation per mode cycle distance. Finally, Tindle and Bold[18] demonstrated that a time displacement in addition to the beam displacement was required to describe the dispersive propagation of wave packets in shallow water.

In this modified ray theory, a ray which is totally reflected at the sea floor is viewed to travel a precise distance along the interface before returning to the water. This beam displacement is a direct consequence of the variation of the phase of the plane-wave reflection coefficient with angle of incidence, and was used by Brekhovskikh[20] to describe the reflection of a bounded acoustic beam exhibiting divergence. A phase change upon reflection which does not vary with angle does not have an associated beam displacement.

The derivation of the beam and time displacement terms is most easily accomplished by again employing the WKB mode functions. The phase of the mode function at the bottom is determined by the plane-wave reflection coefficient at the equivalent ray angle. To show this, we rewrite the WKB mode function near the bottom ($z \lesssim H$):

$$\begin{aligned}\hat{U}_n(z) &\simeq A_n(z)\sin[\phi_n(H)+\gamma_n(H)(z-H)] \\ &= (1/2i)A_n(z)\{\exp[i\phi_n(H)+i\gamma_n(H)(z-H)]\} \\ &\quad + \exp[-i\phi_n(H)-i\gamma_n(H)(z-H)-i\pi]. \end{aligned} \tag{19}$$

Written in this way, the mode function appears as the sum of a plane wave traveling downwards, and a plane wave traveling upwards, with a phase difference. The plane-wave reflection coefficient $R(k_n)$ at the bottom is the complex ratio of the amplitudes at $z = H$:

$$R(k_n) = \exp[-2i\phi_n(H) - i\pi]. \tag{20}$$

The magnitude of the reflection coefficient is obviously one, and the phase $\psi(k_n)$ is equal to $-2\phi_n(H) - \pi$, modulo 2π. This identity is in fact the WKB phase condition:

$$2\phi_n(H) + \psi(k_n) = (2n-1)\pi. \tag{21}$$

The solutions of this equation for successive values of n provide the discrete spectrum of normal mode wavenumbers $\{k_n\}$. Thus if one can specify the phase of the reflection coefficient ψ, the normal mode wavenumbers can be found.

The beam displacement term follows from differentiating this equation with respect to k_n, keeping ω constant. The first term gives the ray skip distance from (15) and the left-hand side gives the mode cycle distance from (7). Rearranging the terms slightly gives

$$\begin{aligned} D_n &= -2\pi\left(\frac{\partial n}{\partial k_n}\right)_\omega \\ &= -2\left(\frac{\partial \phi_n(H)}{\partial k_n}\right)_\omega - \left(\frac{\partial \psi}{\partial k_n}\right)_\omega \\ &= 2c_n^{-1}\int_a^H dz\, c(z)\csc\theta_n(z) + \Delta_n, \end{aligned} \tag{22}$$

in which

$$\Delta_n = -\left(\frac{\partial \psi}{\partial k_n}\right)_\omega. \tag{23}$$

The first quantity on the right-hand side of (22) is obviously the geometric ray skip distance. Thus Δ_n is the beam displacement, that is, that quantity which must be added to the ray skip distance to equal the mode cycle distance. We have ignored the term $-(\partial\phi_n/\partial k_n)_\omega$, which is the beam displacement at the upper turning point considered by Murphy and Davis.[13] It may be included if thought to be significant.

The time displacement associated with the beam displacement follows from differentiating (21) with respect to ω, keeping k_n constant, and employing the chain rule of differentiation to $(\partial n/\partial\omega)_{k_n}$, i.e.,

$$2\left(\frac{\partial \phi_n}{\partial \omega}\right)_{k_n} + \left(\frac{\partial \psi}{\partial \omega}\right)_{k_n} = 2\pi\left(\frac{\partial n}{\partial k_n}\right)_\omega \left(\frac{dk_n}{d\omega}\right) \tag{24}$$

or

$$\frac{D_n}{v_g^{(n)}} = 2\int_a^H dz\, c^{-1}(z)\csc\theta_n(z) + \tau_n \tag{25}$$

in which

$$\tau_n = \left(\frac{\partial \psi}{\partial \omega}\right)_{k_n}. \tag{26}$$

The quantity τ_n is the time displacement associated with the beam displacement Δ_n. Combining (25) with (22) we get an expression for the group velocity of mode n according to modified ray theory:

$$\begin{aligned} v_g^{(n)} &= \left(2c_n^{-1}\int_a^H dz\, c(z)\csc\theta_n(z) + \Delta_n\right) \\ &\quad \times\left(2\int_a^H dz\, c^{-1}(z)\csc\theta_n(z) + \tau_n\right)^{-1}. \end{aligned} \tag{27}$$

Comparing this with expression (5), keeping in mind the results of Sec. IV, it is clear that the beam and time displacements introduced by modified ray theory are equivalent to integrating the mode functions in the bottom.

VI. CONCLUSION

A general formula for normal mode group velocity provides an intuitive method of understanding of the effect of bottom interaction on group velocity. The formula is also

very similar to integrals which calculate mode attenuation, so is easy to implement in existing computer code. By applying WKB theory, the formula reduces in the limit to the average horizontal speed of the ray equivalent to the mode. Moreover, the formula routinely includes all bottom effects in shallow water, effects which require the introduction of beam and time displacements in ray theory.

APPENDIX: THE NORMAL MODE GROUP VELOCITY FORMULA

The analysis presented here is a simplified form of that due to Koch *et al.*,[2] since only compressional waves are considered, whereas the original work contains shear wave effects also. Expression (1) with the boundary conditions (2a) and (2b) gives a discrete spectrum of eigenvalues k_n associated with the unique normal mode functions $U_n(z)$ at a specific value of frequency. If the frequency is changed to $\tilde{\omega}$, the new normal mode functions $\tilde{U}_n(z)$ with wavenumbers $\tilde{k}_n$ are the solutions to

$$\rho(z)\frac{d}{dz}\left(\rho^{-1}(z)\frac{d\tilde{U}_n(z)}{dz}\right)+\left(\frac{\tilde{\omega}^2}{c^2(z)}-\tilde{k}_n^2\right)\tilde{U}_n(z)=0 \tag{A1}$$

with identical boundary conditions (2a) and (2b). Multiplying (1) by $\tilde{U}_n(z)$ and (A1) by $U_n(z)$ and subtracting the products gives

$$\frac{d}{dz}\left(\rho^{-1}(z)\tilde{U}_n(z)\frac{dU_n(z)}{dz}-\rho^{-1}(z)U_n(z)\frac{d\tilde{U}_n(z)}{dz}\right)+\left(\frac{\tilde{\omega}^2-\omega^2}{c^2(z)}-(\tilde{k}_n^2-k_n^2)\right)\rho^{-1}(z)\tilde{U}_n(z)U_n(z)=0. \tag{A2}$$

This expression is integrated from $z=0$ to $z=\infty$, and the first term vanishes due to the boundary conditions, leaving

$$(\tilde{\omega}^2-\omega^2)\int_0^\infty dz\,\rho^{-1}(z)c^{-2}(z)\tilde{U}_n(z)U_n(z)=(\tilde{k}_n^2-k_n^2)\int_0^\infty dz+\rho^{-1}(z)\tilde{U}_n(z)U_n(z). \tag{A3}$$

If $\tilde{\omega}$ is allowed to approach ω infinitesimally, i.e.,

$$\tilde{\omega}=\omega+d\omega, \tag{A4a}$$

$$\tilde{k}_n=k_n+\left(\frac{dk_n}{d\omega}\right)d\omega, \tag{A4b}$$

$$\tilde{U}_n=U_n+\left(\frac{dU_n}{d\omega}\right)d\omega, \tag{A4c}$$

then (A3) can be evaluated to first order in $d\omega$, giving

$$\omega\int_0^\infty dz\,\rho^{-1}(z)c^{-2}(z)U_n^2(z)=k_n\frac{dk_n}{d\omega}\int_0^\infty dz\,\rho^{-1}(z)U_n^2(z). \tag{A5}$$

From the definition of group velocity from (3), this gives the normal mode group velocity,

$$v_g^{(n)}=\int_0^\infty dz\,\rho^{-1}(z)U_n^2(z)\times\left(c_n\int_0^\infty dz\,\rho^{-1}(z)c^{-2}(z)U_n^2(z)\right)^{-1}. \tag{A6}$$

The technique used to derive the derivative $dk_n/d\omega$ is very similar to the technique used to estimate normal mode attenuation coefficients[6] caused by adding a small complex component to $c(z)$.

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A simple, general formula for normal mode group velocities provides an intuitive grasp of the factors influencing group velocity, especially for shallow water environments associated with strong seabed interaction. This is demonstrated in this paper using some hypothetical shallow water environments, and by comparing mode shapes at different frequencies. The formula is especially convenient to implement in normal mode computer codes which already perform similar calculations for mode normalization and attenuation coefficients. Using the WKB approximation to the normal mode functions, the formula is identified with the average horizontal speed of the ray equivalent to the mode. This result strengthens the ray/mode analogy which is used in analysing ocean acoustics problems.

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